mathematics

MDPI

Article

Positive Solutions for Discontinuous Systems via a Multivalued Vector Version of Krasnosel'skiĭ's Fixed Point Theorem in Cones

Rodrigo López Pouso [1], Radu Precup [2,*] and Jorge Rodríguez-López [1]

[1] Departamento de Estatística, Análise Matemática e Optimización, Instituto de Matemáticas, Universidade de Santiago de Compostela, Facultade de Matemáticas, Campus Vida, 15782 Santiago, Spain; rodrigo.lopez@usc.es (R.L.P.); jorgerodriguez.lopez@usc.es (J.R.-L.)

[2] Department of Mathematics, Babeş-Bolyai University, 400084 Cluj, Romania

* Correspondence: r.precup@math.ubbcluj.ro

Received: 12 February 2019; Accepted: 14 May 2019; Published: 20 May 2019

Abstract: We establish the existence of positive solutions for systems of second–order differential equations with discontinuous nonlinear terms. To this aim, we give a multivalued vector version of Krasnosel'skiĭ's fixed point theorem in cones which we apply to a regularization of the discontinuous integral operator associated to the differential system. We include several examples to illustrate our theory.

Keywords: Krasnosel'skiĭ's fixed point theorem; positive solutions; discontinuous differential equations; differential system

1. Introduction

We study the existence and localization of positive solutions for the system

$$\begin{cases} u_1''(t) + g_1(t) f_1(t, u_1(t), u_2(t)) = 0, \\ u_2''(t) + g_2(t) f_2(t, u_1(t), u_2(t)) = 0, \end{cases}$$

subject to the Sturm–Liouville boundary conditions (7).

The novelties in this paper are in two directions. On the one hand, we allow the functions f_i ($i = 1, 2$) to be discontinuous with respect to the unknown over some time-dependent sets, see Definitions 1 and 2. On the other hand, in order to localize the solutions of the system, we shall establish a multivalued vector version of Krasnosel'skiĭ's fixed point theorem which allows different asymptotic behaviors in the nonlinearities f_1 and f_2, see Remark 3.

The existence of discontinuities in the functions f_1 or f_2 makes impossible to apply directly the standard fixed point theorems in cones for compact operators since the integral operator corresponding to the differential problem is not necessarily continuous. In order to avoid this difficulty, we regularize the possibly discontinuous operator obtaining an upper semicontinuous multivalued one. Then we look for fixed points of this multivalued mapping that are proved to be Carathéodory solutions for the differential system. In the case of scalar problems, similar ideas appear in the papers [1–3].

This approach of using set-valued analysis in the study of discontinuous problems is a classical one, see [4]. Nevertheless, the regularization is usually made in the nonlinearities transforming the problem into a differential inclusion and the solutions are often given in the sense of the set-valued analysis (Krasovskij and Filippov solutions [5,6]), see e.g., [7,8]. Similar ideas are also used in the papers [5,9] where there are provided some sufficient conditions for the Krasovskij solutions to be Carathéodory solutions. Recently, second-order scalar discontinuous problems have been investigated by using variational methods [10–12]. However, in these papers there are not considered time-dependent discontinuity sets. Observe also that a lot of existence results for discontinuous differential problems are based on monotonicity hypotheses on their nonlinear parts, see [13], but such assumptions are not necessary in our approach.

Going from scalar discontinuous problems to systems of discontinuous equations is not trivial and it makes possible to consider two different notions for the discontinuity sets. The first approach (see Definition 1 and Theorem 3) allows to study the discontinuities in each variable independently. For instance, it guarantees the existence of a positive solution for the following particular system

$$\begin{cases} -x''(t) = x^2 + x^2y^2H(1-x)H(1-y), \\ -y''(t) = \sqrt{x} + \sqrt{y} + H(x-1)H(y-1), \end{cases}$$

subject to the Sturm–Liouville boundary conditions, where $H : \mathbb{R} \to \mathbb{R}$ is the Heaviside step function given by

$$H(x) = \begin{cases} 0, & \text{if } x \le 0, \\ 1, & \text{if } x > 0, \end{cases}$$

see Example 1. Notice that the nonlinearities in this example are discontinuous at $x = 1$ for each $y \in \mathbb{R}_+$ and at $y = 1$ for every $x \in \mathbb{R}_+$. Moreover, the first nonlinearity has a superlinear behavior and the second one has a sublinear one. Our second approach allows to study functions which are discontinuous over time-dependent curves in $\mathbb{R}^2_+$ and the conditions imposed to these curves are local, see Definition 2 and Theorem 4. In particular, we establish the existence of a positive solution for the system

$$\begin{cases} -x''(t) = (xy)^{1/3}, \\ -y''(t) = \left(1 + (xy)^{1/3}\right) H(x^2 + y^2), \end{cases}$$

subject to the Sturm–Liouville boundary conditions.

As mentioned above, our results rely on fixed point theory for multivalued operators in cones. We finish this introductory part by recalling the version of Krasnosel'skiĭ's fixed point theorem for set-valued maps given by Fitzpatrick–Petryshyn [14].

Theorem 1. *Let X be a Fréchet space with a cone $K \subset X$. Let d be a metric on X and let $r_1, r_2 \in (0, \infty)$, $r = \min\{r_1, r_2\}$, $R = \max\{r_1, r_2\}$ and $F : \overline{B}_R(0) \cap K \longrightarrow 2^K$ usc and condensing. Suppose there exists a continuous seminorm p such that $(I - F)\left(\overline{B}_{r_1}(0) \cap K\right)$ is p-bounded. Moreover, suppose that F satisfies:*

1. *There is some $w \in K$ with $p(w) \neq 0$ and such that $x \notin F(x) + tw$ for any $t > 0$ and $x \in \partial_K B_{r_1}(0)$;*
2. *$\lambda x \notin F(x)$ for any $\lambda > 1$ and $x \in \partial_K B_{r_2}(0)$.*

Then F has a fixed point x_0 with $r \le d(x_0, 0) \le R$.

In the case of a Banach space $(X, \|\cdot\|_X)$ and of an operator $F = (F_1, F_2) : K \subset X^2 \to 2^K$ under the hypotheses of the previous theorem, we obtain the existence of a fixed point $x = (x_1, x_2)$ for F such that $r \le \|x\| \le R$, where $\|\cdot\|$ denotes a norm in X^2, for example, $\|(x_1, x_2)\| = \|x_1\|_X + \|x_2\|_X$. Then $0 \le \|x_1\|_X \le R$ and $0 \le \|x_2\|_X \le R$, but it is not possible to obtain a lower bound for the norm of every component. This fact motivates the use of a vector version of Krasnosel'skiĭ's fixed point theorem. Such a version was introduced in [15] for single-valued operators. Another advantage of the vector approach is that it allows different behaviors in each component of the system.

2. Multivalued Vector Version of Krasnosel'skiĭ's Fixed Point Theorem

In the sequel, let $(X, \|\cdot\|)$ be a Banach space, $K_1, K_2 \subset X$ two cones and $K := K_1 \times K_2$ the corresponding cone of $X^2 = X \times X$. For $r, R \in \mathbb{R}^2_+$, $r = (r_1, r_2)$, $R = (R_1, R_2)$, we denote

$$(K_i)_{r_i,R_i} := \{u \in K_i : r_i \leq \|u\| \leq R_i\} \quad (i = 1,2),$$
$$K_{r,R} := \{u \in K : r_i \leq \|u_i\| \leq R_i \text{ for } i = 1,2\}.$$

The following fixed point theorem is an extension of the vector version of Krasnosel'skiĭ's fixed point theorem given in [15,16] to the class of upper semicontinuous (usc, for short) multivalued mappings.

Theorem 2. *Let $\alpha_i, \beta_i > 0$ with $\alpha_i \neq \beta_i$, $r_i = \min\{\alpha_i, \beta_i\}$ and $R_i = \max\{\alpha_i, \beta_i\}$ for $i = 1,2$. Assume that $N : K_{r,R} \to 2^K$, $N = (N_1, N_2)$, is an usc map with nonempty closed and convex values such that $N(K_{r,R})$ is compact, and there exist $h_i \in K_i \setminus \{0\}$, $i = 1,2$, such that for each $i \in \{1,2\}$ the following conditions are satisfied:*

$$\lambda u_i \notin N_i u \quad \text{for any } u \in K_{r,R} \text{ with } \|u_i\| = \alpha_i \text{ and any } \lambda > 1; \tag{1}$$
$$u_i \notin N_i u + \mu h_i \quad \text{for any } u \in K_{r,R} \text{ with } \|u_i\| = \beta_i \text{ and any } \mu > 0. \tag{2}$$

Then N has a fixed point $u = (u_1, u_2)$ in K, that is, $u \in Nu$, with $r_i \leq \|u_i\| \leq R_i$ for $i = 1,2$.

Proof. We shall consider the four possible combinations of compression-expansion conditions for N_1 and N_2.

1. Assume first that $\beta_i < \alpha_i$ for both $i = 1,2$ (compression for N_1 and N_2). Then $r_i = \beta_i$ and $R_i = \alpha_i$ for $i = 1,2$. Denote $h = (h_1, h_2)$ and define the map $\tilde{N} : K \to K$ given, for $u \in K$, by

$$\tilde{N}u = \min\left\{\frac{\|u_1\|}{r_1}, \frac{\|u_2\|}{r_2}, 1\right\} N\left(\delta_1(u_1)\frac{u_1}{\|u_1\|}, \delta_2(u_2)\frac{u_2}{\|u_2\|}\right) + \left(1 - \min\left\{\frac{\|u_1\|}{r_1}, \frac{\|u_2\|}{r_2}, 1\right\}\right) h,$$

where $\delta_i(u_i) = \max\{\min\{u_i, R_i\}, r_i\}$ for $i = 1,2$.

The map $\tilde{N}$ is usc (the composition of usc maps is usc, see [17], Theorem 17.23) and $\tilde{N}(K)$ is relatively compact since its values belong to the compact set $\overline{\text{co}}\,(N(K_{r,R}) \cup \{h\})$. Then Kakutani's fixed point theorem implies that there exists $u \in K$ such that $u \in \tilde{N}u$.

It remains to prove that $u \in K_{r,R}$. It is clear that $\|u_i\| > 0$ since $h_i \neq 0$ for $i = 1,2$. Assume $0 < \|u_1\| < r_1$ and $0 < \|u_2\| < r_2$. If $\min\left\{\frac{\|u_1\|}{r_1}, \frac{\|u_2\|}{r_2}\right\} = \frac{\|u_1\|}{r_1}$, then

$$u \in \frac{\|u_1\|}{r_1} N\left(\frac{r_1}{\|u_1\|}u_1, \frac{r_2}{\|u_2\|}u_2\right) + \left(1 - \frac{\|u_1\|}{r_1}\right) h,$$

so

$$\frac{r_1}{\|u_1\|}u_1 \in N_1\left(\frac{r_1}{\|u_1\|}u_1, \frac{r_2}{\|u_2\|}u_2\right) + \frac{r_1}{\|u_1\|}\left(1 - \frac{\|u_1\|}{r_1}\right) h_1,$$

what contradicts (2) for $i = 1$. Analogously, we can obtain contradictions for any other point $u \notin K_{r,R}$, as done in [15,16] for single-valued maps.

2. Assume that $\beta_1 < \alpha_1$ (compression for N_1) and $\beta_2 > \alpha_2$ (expansion for N_2). Let $N_i^* : K_{r,R} \to K_i$ $(i = 1, 2)$ be given by

$$\begin{aligned} N_1^* u &= N_1 \left(u_1, \left(\frac{R_2}{\|u_2\|} + \frac{r_2}{\|u_2\|} - 1 \right) u_2 \right), \\ N_2^* u &= \left(\frac{R_2}{\|u_2\|} + \frac{r_2}{\|u_2\|} - 1 \right)^{-1} N_2 \left(u_1, \left(\frac{R_2}{\|u_2\|} + \frac{r_2}{\|u_2\|} - 1 \right) u_2 \right). \end{aligned} \tag{3}$$

   Notice that the map $N^* = (N_1^*, N_2^*)$ is in case 1, and thus N^* has a fixed point $v \in K_{r,R}$. Further, the point u defined as $u_1 = v_1$ and $u_2 = \left(\frac{R_2}{\|v_2\|} + \frac{r_2}{\|v_2\|} - 1 \right) v_2$ is a fixed point of the operator N.
3. The case $\beta_1 > \alpha_1$ (expansion for N_1) and $\beta_2 < \alpha_2$ (compression for N_2) is similar to the previous one by taking the map $N^* = (N_1^*, N_2^*)$ defined as

$$\begin{aligned} N_1^* u &= \left(\frac{R_1}{\|u_1\|} + \frac{r_1}{\|u_1\|} - 1 \right)^{-1} N_1 \left(\left(\frac{R_1}{\|u_1\|} + \frac{r_1}{\|u_1\|} - 1 \right) u_1, u_2 \right), \\ N_2^* u &= N_2 \left(\left(\frac{R_1}{\|u_1\|} + \frac{r_1}{\|u_1\|} - 1 \right) u_1, u_2 \right). \end{aligned} \tag{4}$$

4. The case $\beta_i > \alpha_i$ for $i = 1, 2$ (expansion for N_1 and N_2) reduces to case 1, if we consider the map $N^* = (N_1^*, N_2^*)$ where N_1^* is defined by (4) and N_2^*, by (3).

Therefore, the proof is over. □

Remark 1 (Multiplicity). *Although we are interested in fixed points for the operator N satisfying that both components are nonzero, if we replace conditions* (1) *and* (2) *in Theorem 2 by the following ones:*

$$\begin{aligned} \lambda u_i \notin N_i u \quad &\text{for } \|u_i\| = \alpha_i,\ \|u_j\| \le R_j\ (j \neq i) \text{ and } \lambda \ge 1; \\ u_i \notin N_i u + \mu h_i \quad &\text{for } \|u_i\| = \beta_i,\ \|u_j\| \le R_j\ (j \neq i) \text{ and } \mu \ge 0, \end{aligned}$$

then we can achieve multiplicity results.

Indeed, if $\beta_i > \alpha_i$ for $i = 1$ or $i = 2$, then the operator N has one additional fixed point $v = (v_1, v_2)$ such that $\|v_i\| < r_i$ and $r_j < \|v_j\| < R_j$ with $j \neq i$. Furthermore, if $\beta_i > \alpha_i$ for $i = 1, 2$, then N has three nontrivial fixed points. Such cases are considered in the paper [18] in connection with (p, q)-Laplacian systems.

Our purpose is to apply Theorem 2 to a multivalued regularization of a discontinuous system of single-valued operators associated to a system of differential equations with discontinuous nonlinearities. Our aim is to obtain new existence and localization results for such kind of problems.

In order to do that, we need the following definitions and results.

Let U be a relatively open subset of the cone $K := K_1 \times K_2$ and $T : \overline{U} \to K$, $T = (T_1, T_2)$, an operator not necessarily continuous. We associate to the operator T the following multivalued map $\mathbb{T} : \overline{U} \to 2^K$ given by

$$\mathbb{T} = (\mathbb{T}_1, \mathbb{T}_2), \quad \mathbb{T}_i u = \bigcap_{\varepsilon > 0} \overline{\text{co}}\, T_i \left(\overline{B}_\varepsilon(u) \cap \overline{U} \right) \quad \text{for every } u \in \overline{U} \quad (i = 1, 2), \tag{5}$$

where $\overline{B}_\varepsilon(u) := \{ v \in X^2 : \|u_i - v_i\| \le \varepsilon \text{ for } i = 1, 2 \}$, $\overline{U}$ denotes the closure of the set U with the relative topology of K and $\overline{\text{co}}$ means closed convex hull. The map $\mathbb{T}_i$ is called the closed-convex envelope of T_i and it satisfies the following properties, see [2].

Proposition 1. *Let $\mathbb{T}$ be the closed-convex envelope of an operator $T : \overline{U} \longrightarrow K$. The following properties are satisfied:*

1. *If T maps bounded sets into relatively compact sets, then $\mathbb{T}$ assumes compact values and it is usc;*

2. *If $T\,\overline{U}$ is relatively compact, then $\mathbb{T}\,\overline{U}$ is relatively compact too.*

Remark 2. *The following two statements are equivalent:*

(a) $y \in \mathbb{T}_i(u)$ *$(i = 1, 2)$;*

(b) *for every $\varepsilon > 0$ and every $\rho > 0$ there exist $m \in \mathbb{N}$ and a finite family of vectors $x_j \in \overline{B}_\varepsilon(u) \cap \overline{U}$ and coefficients $\lambda_j \in [0, 1]$ $(j = 1, 2, \ldots, m)$ such that $\sum \lambda_j = 1$ and*

$$\left\| y - \sum_{j=1}^{m} \lambda_j\, T_i x_j \right\| < \rho.$$

3. Positive Solutions of Discontinuous Systems

We study the existence and localization of positive solutions for the following second-order coupled differential system

$$\begin{cases} u_1''(t) + g_1(t) f_1(t, u_1(t), u_2(t)) = 0, \\ u_2''(t) + g_2(t) f_2(t, u_1(t), u_2(t)) = 0, \end{cases} \tag{6}$$

for $t \in I = [0, 1]$, with the following boundary conditions

$$a_i u_i(0) - b_i u_i'(0) = 0, \quad c_i u_i(1) + d_i u_i'(1) = 0, \tag{7}$$

for $i = 1, 2$, where $a_i, b_i, c_i, d_i \in \mathbb{R}_+ \equiv [0, \infty)$ and $\rho_i := b_i c_i + a_i c_i + a_i d_i > 0$ for $i = 1, 2$. Assume that, for $i = 1, 2$,

(H_1) $g_i \in L^1(I)$, $g_i(t) \ge 0$ for a.e. $t \in I$ and $\int_{1/4}^{3/4} g(s)\, ds > 0$;

(H_2) $f_i : I \times \mathbb{R}_+^2 \to \mathbb{R}_+$ satisfies that

- (i) $f_i(\cdot, u_1(\cdot), u_2(\cdot))$ are measurable whenever $(u_1, u_2) \in \mathcal{C}(I)^2$;
- (ii) for each $\rho > 0$ there exists $R_{i,\rho} > 0$ such that

$$f_i(t, u_1, u_2) \le R_{i,r} \quad \text{for } u_1, u_2 \in [0, \rho] \text{ and a.e. } t \in I.$$

Notice that condition (H_2) (i) is satisfied if $f_i(\cdot, u_1, u_2)$ is measurable for all constants u_1, u_2, and if $f_i(t, \cdot, \cdot)$ is continuous for a.a. t, which is not necessarily the case in this paper.

Let $X = \mathcal{C}(I)$ be the space of continuous functions defined on I endowed with the usual norm $\|v\| := \|v\|_\infty = \max_{t \in I} |v(t)|$ and let P be the cone of all nonnegative functions of X. A positive solution to (6)–(7) is a function $u = (u_1, u_2)$ with $u_i \in P \cap W^{2,1}(I)$, $u_i \not\equiv 0$ $(i = 1, 2)$ such that u satisfies (6) for a.a. $t \in I$ and the boundary conditions (7). The existence of positive solutions to problems (6)–(7) is equivalent to the existence of fixed points of the integral operator $T : P^2 \to P^2$, $T = (T_1, T_2)$, given by

$$(T_i u)(t) = \int_0^1 G_i(t, s) g_i(s) f_i(s, u_1(s), u_2(s))\, ds, \qquad i = 1, 2, \tag{8}$$

where $G_i(t, s)$ are the corresponding Green's functions which are explicitly given by

$$G_i(t, s) = \frac{1}{\rho_i} \begin{cases} (c_i + d_i - c_i t)(b_i + a_i s), & \text{if } 0 \le s \le t \le 1, \\ (b_i + a_i t)(c_i + d_i - c_i s), & \text{if } 0 \le t \le s \le 1. \end{cases}$$

Denote

$$M_i := \min\left\{ \frac{c_i + 4d_i}{4(c_i + d_i)}, \frac{a_i + 4b_i}{4(a_i + b_i)} \right\},$$

then it is possible to check the following inequalities:

$$\begin{aligned} G_i(t,s) &\leq G_i(s,s) \quad \text{for } t,s \in I, \\ M_i G_i(s,s) &\leq G_i(t,s) \quad \text{for } t \in [1/4,3/4],\ s \in I. \end{aligned}$$

Consider in X the cones K_1 and K_2 defined as

$$K_i = \left\{ v \in P : v(t) \geq M_i \|v\|_\infty \text{ for all } t \in [1/4,3/4] \right\},$$

and the corresponding cone $K := K_1 \times K_2$ in X^2. Then, $T(K) \subset K$. Indeed, for $u \in K$ and $i = 1,2$,

$$\begin{aligned} M_i \|T_i u\| &= M_i \max_{t\in[0,1]} \int_0^1 G_i(t,s) g_i(s) f_i(s,u_1(s),u_2(s))\, ds \\ &\leq M_i \int_0^1 G_i(s,s) g_i(s) f_i(s,u_1(s),u_2(s))\, ds \leq \min_{t\in[1/4,3/4]} T_i u(t). \end{aligned}$$

Hence, $T_i u \in K_i$ for every $u \in K$ and $i = 1,2$.

Therefore, it must be clear that we intend to apply Theorem 2 in a subset of K to the multivalued operator $\mathbb{T}$ associated to the discontinuous operator T. Later, we shall provide conditions about the functions f_i $(i = 1,2)$ which guarantee that $\mathrm{Fix}(\mathbb{T}) \subset \mathrm{Fix}(T)$, where $\mathrm{Fix}(S)$ stands for the set of fixed points of the mapping S. As a consequence, we obtain some results concerning the existence of positive solutions for system (6)–(7).

Let us introduce some notations. For $\alpha_i, \beta_i > 0$ with $\alpha_i \neq \beta_i$ and $\varepsilon > 0$, we let $r_i = \min\{\alpha_i, \beta_i\}$, $R_i = \max\{\alpha_i, \beta_i\}$ $(i = 1,2)$ and

$$\begin{aligned} f_1^{\beta,\varepsilon} &:= \inf\{f_1(t,u_1,u_2) : t \in [1/4,3/4],\ M_1(\beta_1 - \varepsilon) \leq u_1 \leq \beta_1 + \varepsilon,\ M_2 r_2 \leq u_2 \leq R_2\}, \\ f_2^{\beta,\varepsilon} &:= \inf\{f_2(t,u_1,u_2) : t \in [1/4,3/4],\ M_1 r_1 \leq u_1 \leq R_1,\ M_2(\beta_2 - \varepsilon) \leq u_2 \leq \beta_2 + \varepsilon\}, \\ f_1^{\alpha,\varepsilon} &:= \sup\{f_1(t,u_1,u_2) : t \in [0,1],\ 0 \leq u_1 \leq \alpha_1 + \varepsilon,\ 0 \leq u_2 \leq R_2\}, \\ f_2^{\alpha,\varepsilon} &:= \sup\{f_2(t,u_1,u_2) : t \in [0,1],\ 0 \leq u_1 \leq R_1,\ 0 \leq u_2 \leq \alpha_2 + \varepsilon\}. \end{aligned}$$

Also, denote

$$A_i := \inf_{t\in[1/4,3/4]} \int_{1/4}^{3/4} G_i(t,s) g_i(s)\, ds, \qquad B_i := \sup_{t\in[0,1]} \int_0^1 G_i(t,s) g_i(s)\, ds$$

for $i = 1,2$.

Lemma 1. *Assume that there exist $\alpha_i, \beta_i > 0$ with $\alpha_i \neq \beta_i$, $i = 1,2$, and $\varepsilon > 0$ such that*

$$B_i f_i^{\alpha,\varepsilon} < \alpha_i, \quad A_i f_i^{\beta,\varepsilon} > \beta_i \quad \textit{for } i = 1,2. \tag{9}$$

Then, for each $i \in \{1,2\}$, the following conditions are satisfied:

$$\lambda u_i \notin \mathbb{T}_i u \quad \textit{for any } u \in K_{r,R} \textit{ with } \|u_i\|_\infty = \alpha_i \textit{ and any } \lambda > 1; \tag{10}$$

$$u_i \notin \mathbb{T}_i u + \mu h_i \quad \textit{for any } u \in K_{r,R} \textit{ with } \|u_i\|_\infty = \beta_i \textit{ and any } \mu > 0, \tag{11}$$

where h_1 and h_2 are constant functions equal to 1.

Moreover, the map $\mathbb{T}$ defined as in (5) has at least one fixed point in $K_{r,R}$.

Proof. First, observe that if $v \in K_{r,R}$, then

$$M_i r_i \le v_i(t) \le R_i \quad \text{for all } t \in \left[\frac{1}{4}, \frac{3}{4}\right] \quad (i = 1,2),$$

and if $v \in \overline{B}_\varepsilon(u) \cap K_{r,R}$ for some $u \in K_{r,R}$, and $\|u_1\|_\infty = \alpha_1$, then $v_1(t) \le \alpha_1 + \varepsilon$ for all $t \in [0,1]$ and

$$M_1(\alpha_1 - \varepsilon) \le v_1(t) \le \alpha_1 + \varepsilon \quad \text{for all } t \in \left[\frac{1}{4}, \frac{3}{4}\right].$$

Now we prove (10) for $i = 1$. Assume that $\|u_1\|_\infty = \alpha_1$ and let us see that $\lambda u_1 \notin \mathbb{T}_1 u$ for $\lambda > 1$. First, we shall show that given a family of vectors $v_k \in \overline{B}_\varepsilon(u) \cap K_{r,R}$ and numbers $\lambda_k \in [0,1]$ such that $\sum \lambda_k = 1$ $(k = 1, \dots, m)$, then

$$\lambda u_1 \ne \sum_{k=1}^{m} \lambda_k T_1 v_k,$$

what implies that $\lambda u_1 \notin \operatorname{co}\left(T_1\left(\overline{B}_\varepsilon(u) \cap K_{r,R}\right)\right)$. Indeed, if not, taking the supremum for $t \in [0,1]$,

$$\begin{aligned}
\lambda \alpha_1 &\le \sup_{t\in[0,1]} \sum_{k=1}^{m} \lambda_k \int_0^1 G_1(t,s) g_1(s) f_1(s, v_{k,1}(s), v_{k,2}(s))\, ds \\
&\le \sum_{k=1}^{m} \lambda_k \sup_{t\in[0,1]} \int_0^1 G_1(t,s) g_1(s) f_1(s, v_{k,1}(s), v_{k,2}(s))\, ds \\
&\le \sum_{k=1}^{m} \lambda_k f_1^{\alpha,\varepsilon} B_1 = f_1^{\alpha,\varepsilon} B_1 < \alpha_1,
\end{aligned}$$

a contradiction. Notice that if $\lambda u_1 \in \overline{\operatorname{co}}\left(T_1\left(\overline{B}_\varepsilon(u) \cap K_{r,R}\right)\right)$, then it is the limit of a sequence of functions satisfying the previous inequality and thus, as a limit, it satisfies $\lambda\, \alpha_1 \le \alpha_1$ which is also a contradiction since $\lambda > 1$. Therefore, $\lambda u_1 \notin \mathbb{T}_1 u$ for $\lambda > 1$.

In order to prove (11) for $i = 1$, assume that $\|u_1\|_\infty = \beta_1$ and $u_1 = \sum_{k=1}^{m} \lambda_k T_1 v_k + \mu$ for some family of vectors $v_k \in \overline{B}_\varepsilon(u) \cap K_{r,R}$ and numbers $\lambda_k \in [0,1]$ such that $\sum \lambda_k = 1$ $(k = 1, \dots, m)$ and some $\mu > 0$. Then for $t \in [1/4, 3/4]$, we have

$$\begin{aligned}
u_1(t) &= \sum_{k=1}^{m} \lambda_k \int_0^1 G_1(t,s) g_1(s) f_1(s, v_{k,1}(s), v_{k,2}(s))\, ds + \mu \\
&\ge \sum_{k=1}^{m} \lambda_k \int_{1/4}^{3/4} G_1(t,s) g_1(s) f_1(s, v_{k,1}(s), v_{k,2}(s))\, ds + \mu \\
&\ge \sum_{k=1}^{m} \lambda_k f_{1}^{\beta,\varepsilon} \int_{1/4}^{3/4} G_1(t,s) g_1(s)\, ds + \mu \\
&\ge f_1^{\beta,\varepsilon} A_1 + \mu > \beta_1 + \mu,
\end{aligned}$$

so $\beta_1 > \beta_1 + \mu$, a contradiction. Hence, $u_1 \notin \operatorname{co}\left(T_1\left(\overline{B}_\varepsilon(u) \cap K_{r,R}\right)\right) + \mu h_1$. As before,

$$u_1 \notin \overline{\operatorname{co}}\left(T_1\left(\overline{B}_\varepsilon(u) \cap K_{r,R}\right)\right) + \mu h_1$$

because in that case we arrive to the inequality $\beta_1 \ge \beta_1 + \mu$ for $\mu > 0$. Therefore, $u_1 \notin \mathbb{T}_1(u) + \mu h_1$.

Similarly, it is possible to prove conditions (10) and (11) for $i = 2$.

To finish, the conclusion is obtained by applying Theorem 2 to the operator $\mathbb{T}$. $\square$

Remark 3 (Asymptotic conditions)**.** *The existence of $\alpha_i, \beta_i > 0$ with $\alpha_i \ne \beta_i$, $i = 1,2$, and $\varepsilon > 0$ satisfying (9) is guaranteed, in the autonomous case, by the following sufficient conditions:*

(*a*) *f_1 has a superlinear behavior and f_2, a sublinear one, that is,*

$$\lim_{x\to\infty}\frac{f_1(x,y)}{x}=+\infty \quad \text{for all } y>0, \quad \lim_{x\to 0}\frac{f_1(x,y)}{x}=0 \quad \text{for all } y\geq 0;$$
$$\lim_{y\to\infty}\frac{f_2(x,y)}{y}=0 \quad \text{for all } x\geq 0, \quad \lim_{y\to 0}\frac{f_2(x,y)}{y}=+\infty \quad \text{for all } x>0.$$

(*b*) *Both f_1 and f_2 have a superlinear behavior, that is,*

$$\lim_{x\to\infty}\frac{f_1(x,y)}{x}=+\infty \quad \text{for all } y>0, \quad \lim_{x\to 0}\frac{f_1(x,y)}{x}=0 \quad \text{for all } y\geq 0;$$
$$\lim_{y\to\infty}\frac{f_2(x,y)}{y}=+\infty \quad \text{for all } x>0, \quad \lim_{y\to 0}\frac{f_2(x,y)}{y}=0 \quad \text{for all } x\geq 0.$$

(*c*) *Both f_1 and f_2 have a sublinear behavior, that is,*

$$\lim_{x\to\infty}\frac{f_1(x,y)}{x}=0 \quad \text{for all } y\geq 0, \quad \lim_{x\to 0}\frac{f_1(x,y)}{x}=+\infty \quad \text{for all } y>0;$$
$$\lim_{y\to\infty}\frac{f_2(x,y)}{y}=0 \quad \text{for all } x\geq 0, \quad \lim_{y\to 0}\frac{f_2(x,y)}{y}=+\infty \quad \text{for all } x>0.$$

Remark 4. *If f_1 and f_2 are monotone in both variables, it is possible to specify the numbers $f_i^{\alpha,\varepsilon}$ and $f_i^{\beta,\varepsilon}$ ($i=1,2$), so in this case, conditions (9) only depend on the behavior of the functions at four points in $\mathbb{R}^2_+$, see [15,16].*

Note that Lemma 1 gives us sufficient conditions for the existence of a fixed point in $K_{r,R}$ of the multivalued operator $\mathbb{T}$. Hence, it remains to provide hypothesis on the functions f_i ($i=1,2$) which imply $\mathrm{Fix}(\mathbb{T})\subset \mathrm{Fix}(T)$ in order to obtain a solution for the system (6)–(7). Observe also that no continuity hypotheses were required to the functions f_i until now.

The following definition introduces some curves where we allow the functions f_i to be discontinuous in each variable. The idea of using such curves can be found in some recent papers for second-order discontinuous scalar problems [1–3] and, in some sense, it recalls the notion of time-depending discontinuity sets from [9].

Definition 1. *We say that $\Gamma_1:[a_1,b_1]\subset I=[0,1]\to\mathbb{R}_+$, $\Gamma_1\in W^{2,1}(a_1,b_1)$, is an inviable discontinuity curve with respect to the first variable u_1 if there exist $\varepsilon>0$ and $\psi_1\in L^1(a_1,b_1)$, $\psi_1(t)>0$ for a.e. $t\in[a_1,b_1]$ such that either*

$$\Gamma_1''(t)+\psi_1(t)<-g_1(t)f_1(t,y,z) \text{ for a.e. } t\in[a_1,b_1], \text{ all } y\in[\Gamma_1(t)-\varepsilon,\Gamma_1(t)+\varepsilon] \text{ and all } z\in\mathbb{R}_+, \tag{12}$$

or

$$\Gamma_1''(t)-\psi_1(t)>-g_1(t)f_1(t,y,z) \text{ for a.e. } t\in[a_1,b_1], \text{ all } y\in[\Gamma_1(t)-\varepsilon,\Gamma_1(t)+\varepsilon] \text{ and all } z\in\mathbb{R}_+. \tag{13}$$

Similarly, we say that $\Gamma_2:[a_2,b_2]\subset I=[0,1]\to\mathbb{R}_+$, $\Gamma_2\in W^{2,1}(a_2,b_2)$, is an inviable discontinuity curve with respect to the second variable u_2 if there exist $\varepsilon>0$ and $\psi_2\in L^1(a_2,b_2)$, $\psi_2(t)>0$ for a.e. $t\in[a_2,b_2]$ such that either

$$\Gamma_2''(t)+\psi_2(t)<-g_2(t)f_2(t,y,z) \text{ for a.e. } t\in[a_2,b_2], \text{ all } y\in\mathbb{R}_+ \text{ and all } z\in[\Gamma_2(t)-\varepsilon,\Gamma_2(t)+\varepsilon],$$

or

$$\Gamma_2''(t)-\psi_2(t)>-g_2(t)f_2(t,y,z) \text{ for a.e. } t\in[a_2,b_2], \text{ all } y\in\mathbb{R}_+ \text{ and all } z\in[\Gamma_2(t)-\varepsilon,\Gamma_2(t)+\varepsilon].$$

Now we state some technical results that we need in the proof of the condition $\mathrm{Fix}(\mathbb{T}) \subset \mathrm{Fix}(T)$. Their proofs can be found in [3]. In the sequel, m denotes the Lebesgue measure in $\mathbb{R}$.

Lemma 2 ([3], Lemma 4.1)**.** *Let $a, b \in \mathbb{R}$, $a < b$, and let $g, h \in L^1(a, b)$, $g \geq 0$ a.e., and $h > 0$ a.e. in (a, b). For every measurable set $J \subset (a, b)$ with $m(J) > 0$ there is a measurable set $J_0 \subset J$ with $m(J \setminus J_0) = 0$ such that for every $\tau_0 \in J_0$ we have*

$$\lim_{t\to\tau_0^+} \frac{\int_{[\tau_0,t]\setminus J} g(s)\,ds}{\int_{\tau_0}^t h(s)\,ds} = 0 = \lim_{t\to\tau_0^-} \frac{\int_{[t,\tau_0]\setminus J} g(s)\,ds}{\int_t^{\tau_0} h(s)\,ds}.$$

Corollary 1 ([3], Corollary 4.2)**.** *Let $a, b \in \mathbb{R}$, $a < b$, and let $h \in L^1(a, b)$ be such that $h > 0$ a.e. in (a, b). For every measurable set $J \subset (a, b)$ with $m(J) > 0$ there is a measurable set $J_0 \subset J$ with $m(J \setminus J_0) = 0$ such that for all $\tau_0 \in J_0$ we have*

$$\lim_{t\to\tau_0^+} \frac{\int_{[\tau_0,t]\cap J} h(s)\,ds}{\int_{\tau_0}^t h(s)\,ds} = 1 = \lim_{t\to\tau_0^-} \frac{\int_{[t,\tau_0]\cap J} h(s)\,ds}{\int_t^{\tau_0} h(s)\,ds}.$$

We shall also need the following lemma, see [2], Lemma 3.11.

Lemma 3. *If $M \in L^1(0, 1)$, $M \geq 0$ almost everywhere, then the set*

$$Q = \left\{ u \in \mathcal{C}^1([0,1]) : \left|u'(t) - u'(s)\right| \leq \int_s^t M(r)\,dr \quad \text{whenever } 0 \leq s \leq t \leq 1 \right\}$$

is closed in $\mathcal{C}([0, 1])$ endowed with the maximum norm topology.

Moreover, if $u_n \in Q$ for all $n \in \mathbb{N}$ and $u_n \to u$ uniformly in $[0, 1]$, then there exists a subsequence $\{u_{n_k}\}$ which tends to u in the $\mathcal{C}^1$ norm.

Now we are ready to present the following existence and localization result for the differential system (6)–(7).

Theorem 3. *Suppose that the functions f_i and g_i $(i = 1, 2)$ satisfy conditions (H_1), (H_2) and*

(H_3) *There exist inviable discontinuity curves $\Gamma_{1,n} : I_{1,n} := [a_{1,n}, b_{1,n}] \subset I \to \mathbb{R}_+$ with respect to the first variable, $n \in \mathbb{N}$, and inviable discontinuity curves $\Gamma_{2,n} : I_{2,n} := [a_{2,n}, b_{2,n}] \subset I \to \mathbb{R}_+$ with respect to the second variable, $n \in \mathbb{N}$, such that for each $i \in \{1, 2\}$ and for a.e. $t \in I$ the function $(u_1, u_2) \mapsto f_i(t, u_1, u_2)$ is continuous on*

$$\left(\mathbb{R}_+ \setminus \bigcup_{\{n: t\in I_{1,n}\}} \{\Gamma_{1,n}(t)\} \right) \times \left(\mathbb{R}_+ \setminus \bigcup_{\{n: t\in I_{2,n}\}} \{\Gamma_{2,n}(t)\} \right).$$

Moreover, assume that there exist $\alpha_i, \beta_i > 0$ with $\alpha_i \neq \beta_i$, $i = 1, 2$, and $\varepsilon > 0$ such that

$$B_i f_i^{\alpha,\varepsilon} < \alpha_i, \quad A_i f_i^{\beta,\varepsilon} > \beta_i \quad \text{for } i = 1, 2.$$

Then system (6)–(7) has at least one solution in $K_{r,R}$.

Proof. The operator $T : K_{r,R} \to K$, $T = (T_1, T_2)$, given by (8) is well-defined and the hypotheses (H_1) and (H_2) imply that $T(K_{r,R})$ is relatively compact as an immediate consequence of the

Ascoli–Arzelá theorem. Moreover, by (H_1) and (H_2), there exist functions $\eta_i \in L^1(I)$ $(i = 1, 2)$ such that

$$g_i(t) f_i(t, u_1, u_2) \le \eta_i(t) \quad \text{for a.e. } t \in I \text{ and all } u_1 \in [0, R_1],\ u_2 \in [0, R_2]. \tag{14}$$

Therefore, $T(K_{r,R}) \subset Q_1 \times Q_2$, where

$$Q_i = \left\{ u \in \mathcal{C}^1([0,1]) : |u'(t) - u'(s)| \le \int_s^t \eta_i(r)\, dr \quad \text{whenever } 0 \le s \le t \le 1 \right\},$$

for $i = 1, 2$, which by virtue of Lemma 3 is a closed and convex subset of $X = \mathcal{C}(I)$. Then, by 'convexification', $\mathbb{T}(K_{r,R}) \subset Q_1 \times Q_2$, where $\mathbb{T}$ is the multivalued map associated to T defined as in (5).

By Lemma 1, the multivalued map $\mathbb{T}$ has a fixed point in $K_{r,R}$. Hence, if we show that all the fixed points of the operator $\mathbb{T}$ are fixed points of T, the conclusion is obtained. To do so, we fix an arbitrary function $u \in K_{r,R} \cap (Q_1 \times Q_2)$ and we consider three different cases.

Case 1: $m(\{t \in I_{1,n} : u_1(t) = \Gamma_{1,n}(t)\} \cup \{t \in I_{2,n} : u_2(t) = \Gamma_{2,n}(t)\}) = 0$ for all $n \in \mathbb{N}$. Let us prove that T is continuous at u, which implies that $\mathbb{T}u = \{Tu\}$, and therefore the relation $u \in \mathbb{T}u$ gives that $u = Tu$.

The assumption implies that for a.a. $t \in I$ the mappings $f_1(t, \cdot)$ and $f_2(t, \cdot)$ are continuous at $u(t) = (u_1(t), u_2(t))$. Hence if $u_k \to u$ in $K_{r,R}$ then

$$f_i(t, u_k(t)) \to f_i(t, u(t)) \quad \text{for a.a. } t \in I \text{ and for } i = 1, 2,$$

which, along with (14), yield $Tu_k \to Tu$ in $\mathcal{C}(I)^2$, so T is continuous at u.

Case 2: $m(\{t \in I_{1,n} : u_1(t) = \Gamma_{1,n}(t)\}) > 0$ for some $n \in \mathbb{N}$. In this case we can prove that $u_1 \notin \mathbb{T}_1 u$, and thus $u \notin \mathbb{T}u$.

To this aim, first, we fix some notation. Let us assume that for some $n \in \mathbb{N}$ we have $m(\{t \in I_{1,n} : u_1(t) = \Gamma_{1,n}(t)\}) > 0$ and there exist $\varepsilon > 0$ and $\psi \in L^1(I_{1,n})$, $\psi(t) > 0$ for a.a. $t \in I_{1,n}$, such that (13) holds with Γ_1 replaced by $\Gamma_{1,n}$. (The proof is similar if we assume (12) instead of (13), so we omit it.)

We denote $J = \{t \in I_{1,n} : u_1(t) = \Gamma_{1,n}(t)\}$, and we deduce from Lemma 2 that there is a measurable set $J_0 \subset J$ with $m(J_0) = m(J) > 0$ such that for all $\tau_0 \in J_0$ we have

$$\lim_{t \to \tau_0^+} \frac{2 \int_{[\tau_0, t] \setminus J} \eta_1(s)\, ds}{(1/4) \int_{\tau_0}^t \psi(s)\, ds} = 0 = \lim_{t \to \tau_0^-} \frac{2 \int_{[t, \tau_0] \setminus J} \eta_1(s)\, ds}{(1/4) \int_t^{\tau_0} \psi(s)\, ds}. \tag{15}$$

By Corollary 1 there exists $J_1 \subset J_0$ with $m(J_0 \setminus J_1) = 0$ such that for all $\tau_0 \in J_1$ we have

$$\lim_{t \to \tau_0^+} \frac{\int_{[\tau_0, t] \cap J_0} \psi(s)\, ds}{\int_{\tau_0}^t \psi(s)\, ds} = 1 = \lim_{t \to \tau_0^-} \frac{\int_{[t, \tau_0] \cap J_0} \psi(s)\, ds}{\int_t^{\tau_0} \psi(s)\, ds}. \tag{16}$$

Let us now fix a point $\tau_0 \in J_1$. From (15) and (16) we deduce that there exist $t_- < \tilde{t}_- < \tau_0$ and $t_+ > \tilde{t}_+ > \tau_0$, $t_\pm$ sufficiently close to τ_0 so that the following inequalities are satisfied for all $t \in [\tilde{t}_+, t_+]$:

$$2 \int_{[\tau_0, t] \setminus J} \eta_1(s)\, ds < \frac{1}{4} \int_{\tau_0}^t \psi(s)\, ds, \tag{17}$$

$$\int_{[\tau_0, t] \cap J} \psi(s)\, ds \ge \int_{[\tau_0, t] \cap J_0} \psi(s)\, ds > \frac{1}{2} \int_{\tau_0}^t \psi(s)\, ds, \tag{18}$$

and for all $t \in [t_-, \tilde{t}_-]$:

$$2\int_{[t,\tau_0]\setminus J} \eta_1(s)\,ds < \frac{1}{4}\int_t^{\tau_0} \psi(s)\,ds, \tag{19}$$

$$\int_{[t,\tau_0]\cap J} \psi(s)\,ds > \frac{1}{2}\int_t^{\tau_0} \psi(s)\,ds. \tag{20}$$

Finally, we define a positive number

$$\tilde{\rho} = \min\left\{\frac{1}{4}\int_{\tilde{t}_-}^{\tau_0} \psi(s)\,ds, \frac{1}{4}\int_{\tau_0}^{\tilde{t}_+} \psi(s)\,ds\right\}, \tag{21}$$

and we are ready to prove that $u_1 \notin \mathbb{T}_1 u$. It suffices to prove the following claim:

Claim: let $\varepsilon > 0$ be given by our assumptions over $\Gamma_{1,n}$ as Definition 1 shows, and let $\rho = \frac{\tilde{\rho}}{2}\min\{\tilde{t}_- - t_-, t_+ - \tilde{t}_+\}$, where $\tilde{\rho}$ is as in (21). For every finite family $x_j \in \overline{B}_\varepsilon(u) \cap K_{r,R}$ and $\lambda_j \in [0,1]$ ($j = 1, 2, \ldots, m$), with $\sum \lambda_j = 1$, we have $\|u_1 - \sum \lambda_j T_1 x_j\|_\infty \geq \rho$.

Let x_j and λ_j be as in the Claim and, for simplicity, denote $y = \sum \lambda_j T_1 x_j$. For a.a. $t \in J = \{t \in I_{1,n} \,:\, u_1(t) = \Gamma_{1,n}(t)\}$ we have

$$y''(t) = \sum_{j=1}^{m} \lambda_j (T_1 x_j)''(t) = -\sum_{j=1}^{m} \lambda_j\, g_1(t) f_1(t, x_{j,1}(t), x_{j,2}(t)). \tag{22}$$

On the other hand, for every $j \in \{1, 2, \ldots, m\}$ and every $t \in J$ we have

$$|x_{j,1}(t) - \Gamma_{1,n}(t)| = |x_{j,1}(t) - u_1(t)| < \varepsilon,$$

and then the assumptions on $\Gamma_{1,n}$ ensure that for a.a. $t \in J$ we have

$$y''(t) = -\sum_{j=1}^{m} \lambda_j\, g_1(t) f_1(t, x_{j,1}(t), x_{j,2}(t)) < \sum_{j=1}^{m} \lambda_j\,(\Gamma''_{1,n}(t) - \psi(t)) = u_1''(t) - \psi(t). \tag{23}$$

Now for $t \in [t_-, \tilde{t}_-]$ we compute

$$\begin{aligned}
y'(\tau_0) - y'(t) &= \int_t^{\tau_0} y''(s)\,ds = \int_{[t,\tau_0]\cap J} y''(s)\,ds + \int_{[t,\tau_0]\setminus J} y''(s)\,ds \\
&< \int_{[t,\tau_0]\cap J} u_1''(s)\,ds - \int_{[t,\tau_0]\cap J} \psi(s)\,ds \\
&\quad + \int_{[t,\tau_0]\setminus J} \eta_1(s)\,ds \qquad \text{(by (23), (22) and (14))} \\
&= u_1'(\tau_0) - u_1'(t) - \int_{[t,\tau_0]\setminus J} u_1''(s)\,ds - \int_{[t,\tau_0]\cap J} \psi(s)\,ds + \int_{[t,\tau_0]\setminus J} \eta_1(s)\,ds \\
&\leq u_1'(\tau_0) - u_1'(t) - \int_{[t,\tau_0]\cap J} \psi(s)\,ds + 2\int_{[t,\tau_0]\setminus J} \eta_1(s)\,ds \\
&< u_1'(\tau_0) - u_1'(t) - \frac{1}{4}\int_t^{\tau_0} \psi(s)\,ds \quad \text{(by (19) and (20))},
\end{aligned}$$

hence $y'(t) - u_1'(t) \geq \tilde{\rho}$ provided that $y'(\tau_0) \geq u_1'(\tau_0)$. Therefore, by integration we obtain

$$y(\tilde{t}_-) - u_1(\tilde{t}_-) = y(t_-) - u_1(t_-) + \int_{t_-}^{\tilde{t}_-} (y'(t) - u_1'(t))\,dt \geq y(t_-) - u_1(t_-) + \tilde{\rho}(\tilde{t}_- - t_-).$$

So, if $y(t_-) - u_1(t_-) \leq -\rho$, then $\|y - u_1\|_\infty \geq \rho$. Otherwise, if $y(t_-) - u_1(t_-) > -\rho$, then we have $y(\tilde{t}_-) - u_1(\tilde{t}_-) > \rho$ and thus $\|y - u_1\|_\infty \geq \rho$, too.

Similar computations in the interval $[\tilde{t}_+, t_+]$ instead of $[t_-, \tilde{t}_-]$ show that if $y'(\tau_0) \leq u_1'(\tau_0)$ then we have $u_1'(t) - y'(t) \geq \tilde{\rho}$ for all $t \in [\tilde{t}_+, t_+]$ and this also implies $\|y - u_1\| \geq \rho$. The claim is proven.

Case 3: $m(\{t \in I_{2,n} \,:\, u_2(t) = \Gamma_{2,n}(t)\}) > 0$ for some $n \in \mathbb{N}$. In this case it is possible to prove that $u_2 \notin \mathbb{T}_2 u$. The details are similar to those in Case 2, with obvious changes, so we omit them. □

Remark 5. *Observe that Definition 1 allows to study the discontinuities of the functions f_i independently in each variable u_1 and u_2, as shown in condition (H_3).*

In addition, a continuum set of discontinuity points is possible: for instance, the function f_1 may be discontinuous at the point $u_1 = 1$ for all $u_2 \in \mathbb{R}_+$ provided that the constant function $\Gamma_1 \equiv 1$ is an inviable discontinuity curve with respect to the first variable. This fact improves the ideas given in [5] for first-order autonomous systems where "only" a countable set of discontinuity points are allowed.

Remark 6. *Notice that conditions (12) and (13) are not local in the last variable. However, the condition*

$$\inf_{t \in I, x, y \in \mathbb{R}_+} f_1(t, x, y) > 0$$

implies that any constant function stands for an inviable discontinuity curve with respect to the first variable (since condition (13) holds). Moreover, any function with strictly positive second derivative is always an inviable discontinuity curve with respect to the variable u_1 without any additional condition on f_1.

Now we illustrate our existence result by some examples.

Example 1. *Consider the coupled system*

$$\begin{cases} -x''(t) = x^2 + x^2 y^2 H(a - x) H(b - y), \\ -y''(t) = \sqrt{x} + \sqrt{y} + H(x - c) H(y - d), \end{cases} \tag{24}$$

subject to the boundary conditions (7) (replacing u_1 and u_2 by x and y, respectively) where $a, b, c, d > 0$ and H denotes the Heaviside function.

The existence of numbers α_i and β_i in the conditions of (9) is guaranteed by Remark 3 (a) since $f_1(x, y) = x^2 + x^2 y^2 H(a - x) H(b - y)$ is a superlinear function and $f_2(x, y) = \sqrt{x} + \sqrt{y} + H(x - c) H(y - d)$ is a sublinear function.

On the other hand, the function $(x, y) \mapsto f_1(x, y)$ is continuous on $(\mathbb{R}_+ \setminus \{a\}) \times (\mathbb{R}_+ \setminus \{b\})$ and the constant function $\Gamma_1 \equiv a$ stands for an inviable curve with respect to the first variable. Indeed,

$$-\Gamma_1''(t) + \frac{a^2}{8} = \frac{a^2}{8} < f_1(y, z) \quad \text{for a.a. } t \in [0, 1] \text{ and for all } y \in \left[\frac{a}{2}, \frac{3a}{2}\right] \text{ and } z \in \mathbb{R}_+,$$

hence (13) holds with $\psi_1 \equiv a^2/8$.

Moreover, the constant function $\Gamma_2 \equiv b$ is an inviable curve with respect to the second variable, according to Remark 6 since

$$\inf_{x, y \in \mathbb{R}_+} f_2(x, y) > 0.$$

Similarly, the function $f_2(x, y) = \sqrt{x} + \sqrt{y} + H(x - c) H(y - d)$ satisfies the hypothesis (H_3) in Theorem 3, so the system (7)–(24) has at least one positive solution.

Example 2. *Consider the system*

$$\begin{cases} -x''(t) = x^2 + x^2 y^2 H(a + t^2 - x) H(b + mt - y), \\ -y''(t) = \sqrt{x} + \sqrt{y} + H(x - c) H(y - d), \end{cases} \tag{25}$$

subject to the boundary conditions (7), where $a, b, c, d > 0$ and $m \in \mathbb{R}$.

Now, for a.a. $t \in I$, the function $(x, y) \mapsto f_1(t, x, y)$, where

$$f_1(t, x, y) = x^2 + x^2y^2H(a + t^2 - x)H(b + mt - y),$$

is continuous on $\left(\mathbb{R}_+ \setminus \{a + t^2\}\right) \times (\mathbb{R}_+ \setminus \{b + mt\})$ and the curve $\Gamma_1(t) = a + t^2$ is inviable with respect to the first variable. Indeed, (13) is satisfied with $\psi_1 \equiv 1$, since

$$-\Gamma_1''(t) + 1 = -1 < f_1(t, y, z) \quad \text{for a.a. } t \in [0, 1] \text{ and for all } y, z \in \mathbb{R}_+.$$

On the other hand, the curve $\Gamma_2(t) = b + mt$ is inviable with respect to the variable y, according to Remark 6, since $\Gamma_2''(t) \equiv 0$ and $\inf_{x,y\in\mathbb{R}_+} f_2(x, y) > 0$.

Therefore, Theorem 3 ensures the existence of one positive solution for problem (7)–(25).

Nevertheless, the conditions of Definition 1 are too strong for functions f_1 which are discontinuous at a single isolated point (x_0, y_0) or, more generally, over a curve $(\gamma_1(t), \gamma_2(t))$ for $t \in \bar{I} \subset I$. This is the motivation for another definition of the notion of discontinuity curves. This notion will be a generalization of the admissible curves presented in [2] for one equation.

Definition 2. *We say that $\gamma = (\gamma_1, \gamma_2) : [a, b] \subset I = [0, 1] \to \mathbb{R}^2_+$, $\gamma_i \in W^{2,1}(a, b)$ $(i = 1, 2)$, is an admissible discontinuity curve for the differential equation $u_1'' = -g_1(t)f_1(t, u_1(t), u_2(t))$ if one of the following conditions holds:*

(a) $\gamma_1''(t) = -g_1(t)f_1(t, \gamma_1(t), \gamma_2(t))$ *for a.e. $t \in [a, b]$ (then we say γ is viable for the differential equation),*

(b) *There exist $\varepsilon > 0$ and $\psi \in L^1(a, b)$, $\psi(t) > 0$ for a.e. $t \in [a, b]$ such that either*

$$\gamma_1''(t) + \psi(t) < -g_1(t)f_1(t, y, z) \quad \text{for a.e. } t \in [a, b] \text{ all } y \in [\gamma_1(t) - \varepsilon, \gamma_1(t) + \varepsilon] \text{ and all } z \in [\gamma_2(t) - \varepsilon, \gamma_2(t) + \varepsilon]\,,$$

or

$$\gamma_1''(t) - \psi(t) > -g_1(t)f_1(t, y, z) \quad \text{for a.e. } t \in [a, b] \text{ all } y \in [\gamma_1(t) - \varepsilon, \gamma_1(t) + \varepsilon] \text{ and all } z \in [\gamma_2(t) - \varepsilon, \gamma_2(t) + \varepsilon]\,.$$

In this case we say that γ is inviable.

Similarly, we can define admissible discontinuity curves for $u_2'' = -g_2(t)f_2(t, u_1(t), u_2(t))$.

Theorem 4. *Suppose that the functions f_i and g_i $(i = 1, 2)$ satisfy conditions (H_1), (H_2) and*

(H_3^*) *There exist admissible discontinuity curves for the first differential equation $\gamma_n : I_n := [a_n, b_n] \to \mathbb{R}^2_+$, $n \in \mathbb{N}$, such that for a.e. $t \in I$ the function $(u_1, u_2) \mapsto f_1(t, u_1, u_2)$ is continuous on $\mathbb{R}^2_+ \setminus \bigcup_{\{n : t\in I_n\}} \{(\gamma_{n,1}(t), \gamma_{n,2}(t))\}$;*

(H_4^*) *There exist admissible discontinuity curves for the second differential equation $\tilde{\gamma}_n : \tilde{I}_n := [\tilde{a}_n, \tilde{b}_n] \to \mathbb{R}^2_+$, $n \in \mathbb{N}$, such that for a.e. $t \in I$ the function $(u_1, u_2) \mapsto f_2(t, u_1, u_2)$ is continuous on $\mathbb{R}^2_+ \setminus \bigcup_{\{n : t\in \tilde{I}_n\}} \{(\tilde{\gamma}_{n,1}(t), \tilde{\gamma}_{n,2}(t))\}$.*

Moreover, assume that there exist $\alpha_i, \beta_i > 0$ with $\alpha_i \neq \beta_i$, $i = 1, 2$, and $\varepsilon > 0$ such that

$$B_i f_i^{\alpha,\varepsilon} < \alpha_i, \quad A_i f_i^{\beta,\varepsilon} > \beta_i \quad \text{for } i = 1, 2.$$

Then the differential system (6)–(7) has at least one solution in $K_{r,R}$.

Proof. Notice that in virtue of Lemma 1 it is sufficient to show that $\text{Fix}(\mathbb{T}) \subset \text{Fix}(T)$. Reasoning as in the proof of Theorem 3, if we fix a function $u \in K_{r,R} \cap (Q_1 \times Q_2)$, we have to consider three different cases.

Case 1: $m(\{t \in I_n : u(t) = \gamma_n(t)\} \cup \{t \in \tilde{I}_n : u(t) = \tilde{\gamma}_n(t)\}) = 0$ for all $n \in \mathbb{N}$. Then T is continuous at u.

Case 2: $m(\{t \in I_n : u(t) = \gamma_n(t)\}) > 0$ or $m(\{t \in \tilde{I}_n : u(t) = \tilde{\gamma}_n(t)\}) > 0$ for some γ_n or $\tilde{\gamma}_n$ inviable. Then $u \notin \mathbb{T}u$. The proof follows the ideas from Case 2 in Theorem 3.

Case 3: $m(\{t \in I_n : u(t) = \gamma_n(t)\}) > 0$ or $m(\{t \in \tilde{I}_n : u(t) = \tilde{\gamma}_n(t)\}) > 0$ only for viable curves. Then the relation $u \in \mathbb{T}u$ implies $u = Tu$. In this case the idea is to show that u is a solution of the differential system. The proof is analogus to that of the equivalent case in [2], Theorem 3.12 or [3], Theorem 4.4, so we omit it here. □

Remark 7. *Notice that, in the case of a function $(u_1, u_2) \mapsto f_1(t, u_1, u_2)$ which is discontinuous at a single point (x_0, y_0), Definition 2 requires that one of the following two conditions holds:*

- (i) $f_1(t, x_0, y_0) = 0$ *for a.e.* $t \in [0, 1]$;
- (ii) *there exist $\varepsilon > 0$ and $\psi \in L^1(0, 1)$, $\psi(t) > 0$ for a.e. $t \in I$ such that*

$$0 < \psi(t) < g_1(t) f_1(t, x, y) \text{ for a.e. } t \in I, \text{ all } x \in [x_0 - \varepsilon, x_0 + \varepsilon] \text{ and all } y \in [y_0 - \varepsilon, y_0 + \varepsilon].$$

In particular, for (ii), it suffices that there exist $\varepsilon, \delta > 0$ such that

$$0 < \delta < f_1(t, x, y) \text{ for a.e. } t \in I, \text{ all } x \in [x_0 - \varepsilon, x_0 + \varepsilon] \text{ and all } y \in [y_0 - \varepsilon, y_0 + \varepsilon].$$

To finish, we present two simple examples which fall outside of the applicability of Theorem 3, but which can be studied by means of Theorem 4.

Example 3. *Consider the problem*

$$\begin{cases} -x''(t) = f_1(x, y) = (xy)^{1/3} \left(2 - \cos\left(1/((x-1)^2 + (y-1)^2)\right) H\left((x-1)^2 + (y-1)^2\right)\right), \\ -y''(t) = f_2(x, y) = (xy)^{1/3}, \end{cases} \tag{26}$$

subject to the boundary conditions (7).

It is clear that f_1 and f_2 have a sublinear behavior, see Remark 3.

The function $(x, y) \mapsto f_1(x, y)$ is continuous on $\mathbb{R}^2_+ \setminus \{(1, 1)\}$ and the constant function $\gamma(t) = (\gamma_1(t), \gamma_2(t)) \equiv (1, 1)$ is an inviable admissible discontinuity curve for the differential equation $-x''(t) = f_1(x, y)$ since $0 < 1/\sqrt[3]{4} \le f_1(x, y)$ for all $x \in [1/2, 3/2]$ and all $y \in [1/2, 3/2]$; and $\gamma_1''(t) = 0$.

Therefore, Theorem 4 guarantees the existence of a positive solution for problem (7)–(26).

Example 4. *Consider the following system*

$$\begin{cases} -x''(t) = f_1(x, y) = (xy)^{1/3}, \\ -y''(t) = f_2(x, y) = \left(1 + (xy)^{1/3}\right) H(x^2 + y^2), \end{cases} \tag{27}$$

subject to the boundary conditions (7).

The nonlinearities of the system have again a sublinear behavior. Now, the function $(x, y) \mapsto f_2(x, y)$ is continuous on $\mathbb{R}^2_+ \setminus \{(0, 0)\}$ and the constant function $\gamma(t) = (\gamma_1(t), \gamma_2(t)) \equiv (0, 0)$ is a viable admissible discontinuity curve for the differential equation.

Hence, by application of Theorem 4, one obtains that the system (7)–(27) has at least one positive solution.

Author Contributions: The authors contributed equally to this work.

Funding: R. López Pouso was partially supported by Ministerio de Economía y Competitividad, Spain, and FEDER, Project MTM2016-75140-P, and Xunta de Galicia ED341D R2016/022 and GRC2015/004. Jorge Rodríguez-López was partially supported by Xunta de Galicia Scholarship ED481A-2017/178.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Figueroa, R.; Infante, G. A Schauder–type theorem for discontinuous operators with applications to second-order BVPs. *Fixed Point Theory Appl.* **2016**, *2016*, 53. [CrossRef]
2. Figueroa, R.; Pouso, R.L.; Rodríguez-López, J. A version of Krasnosel'skiĭ's compression-expansion fixed point theorem in cones for discontinuous operators with applications. *Topol. Methods Nonlinear Anal.* **2018**, *51*, 493–510. [CrossRef]
3. López Pouso, R. Schauder's fixed–point theorem: New applications and a new version for discontinuous operators. *Bound. Value Probl.* **2012**, *2012*, 92. [CrossRef]
4. Filippov, A.F. *Differential Equations with Discontinuous Righthand Sides*; Kluwer Academic: Dordrecht, The Netherlands, 1988.
5. Hu, S. Differential equations with discontinuous right-hand sides. *J. Math. Anal. Appl.* **1991**, *154*, 377–390. [CrossRef]
6. Spraker, J.S.; Biles, D. A comparison of the Carathéodory and Filippov solution sets. *J. Math. Anal. Appl.* **1996**, *198*, 571–580. [CrossRef]
7. Bereanu, C.; Jebelean, P.; Şerban, C. The Dirichlet problem for discontinuous perturbations of the mean curvature operator in Minkowski space. *Electron. J. Qual. Theory Differ. Equ.* **2015**, *35*, 1–7. [CrossRef]
8. Bonanno, G.; Jebelean, P.; Şerban, C. Three periodic solutions for discontinuous perturbations of the vector p-Laplacian operator. *Proc. R. Soc. Edinb. A* **2017**, *147*, 673–681. [CrossRef]
9. Cid, J.A.; Pouso, R.L. Ordinary differential equations and systems with time-dependent discontinuity sets. *Proc. R. Soc. Edinb. A* **2004**, *134*, 617–637. [CrossRef]
10. Bonanno, G.; Bisci, G.M. Infinitely many solutions for a boundary value problem with discontinuous nonlinearities. *Bound. Value Probl.* **2009**, *2009*, 670675. [CrossRef]
11. Bonanno, G.; Buccellato, S.M. Two point boundary value problems for the Sturm–Liouville equation with highly discontinuous nonlinearities. *Taiwan J. Math.* **2010**, *14*, 2059–2072. [CrossRef]
12. Bonanno, G.; Iannizzotto, A.; Marras, M. On ordinary differential inclusions with mixed boundary conditions. *Differ. Integral Equ.* **2017**, *30*, 273–288.
13. Heikkilä, S.; Lakshmikantham, V. *Monotone Iterative Techniques for Discontinuous Nonlinear Differential Equations*; Marcel Dekker: New York, NY, USA, 1994.
14. Fitzpatrick, P.M.; Petryshyn, W.V. Fixed point theorems and the fixed point index for multivalued mappings in cones. *J. Lond. Math. Soc.* **1975**, *2*, 75–85. [CrossRef]
15. Precup, R. A vector version of Krasnosel'skiĭ's fixed point theorem in cones and positive periodic solutions of nonlinear systems. *J. Fixed Point Theory Appl.* **2007**, *2*, 141–151. [CrossRef]
16. Precup, R. Componentwise compression-expansion conditions for systems of nonlinear operator equations and applications. In *Mathematical Models in Engineering, Biology and Medicine*; AIP Conference Proceedings; American Institute of Physics: Melville, NY, USA, 2009; Volume 1124, pp. 284–293.
17. Aliprantis, C.D.; Border, K.C. *Infinite Dimensional Analysis. A Hitchhiker's Guide*, 3rd ed.; Springer: New York, NY, USA, 2006.
18. Infante, G.; Maciejewski, M.; Precup, R. A topological approach to the existence and multiplicity of positive solutions of (p, q)-Laplacian systems. *Dyn. Partial Differ. Equ.* **2015**, *12*, 193–215. [CrossRef]